Network Design Problems with Two-Edges Matching Failures

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In this paper, we consider a problem named Network Design Problems with two-edges matching failures (NDP2EM). In a graph, any two non-incident edges is a two-edges matching. We ask for a minimum cost subgraph that contains at least one path between any pair of terminal nodes after deleting any two-edges matchings from the graph. We develop a formulation of the NDP2EM and show that how one can effectively compute the lower bound for the problem.

Introduction

Connected network design problems come up naturally when we analyze the connectivity requirements for a network, which has to be built to be able to flow not only prescribed traffic requirements in normal operation, but also the same amount of traffic when failures occur on some components (edges or vertices). Another reason for the growth of interest in this area is the variety of contexts of designing transportation networks, telecommunication networks and teleprocessing networks.

In papers [1]-[4], the authors studied on telecommunication network design and they have focused on k-connectivity ($k \geq 2$) of the constrained network problem, in which one wants to design a network with the minimum prescribed number of disjoint paths between any pair of nodes. The polyhedra associated with these problems have been studied in [3]-[6].

The network design problems with two-edges matching failures (NDP2EM) consists of finding a minimum cost subgraph G_* which contains at least one path between every pair of terminal nodes of N after deleting any non-incident two edges from the graph G_*.

Model of the NDP2EM

In order to formulate the NDP2EM on an undirected network G with given terminal nodes in N, we fix any node s in N as the source and add a new node r to the network G as the sink. Then all terminal nodes t in $N_0 = N \setminus s$ are connected by the arcs (t, r) with unit capacities, i.e., the edge t, r is directed from the node t to node r. Then all edges $s, v \in E$ are directed from s to v with capacity b, where $v \in V$ and $b = |N_0|$. Lastly, each edges $v, w \in E$ where $v \neq s, r$ is replaced by two arcs with opposite directions. All these arcs have capacity b. Let Π be the set all two-edges matchings in G. To illustrate the formulation of the problem let us consider the NDP2EM on a complete graph with four nodes (Figure 1.(a)), where nodes 1,2 and 3 are terminal nodes.

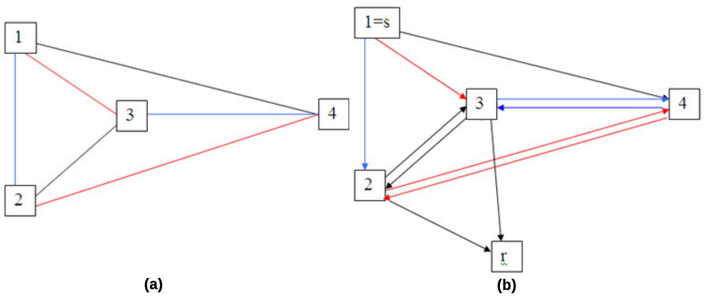

Figure 1. (a) original network (b) modified network

Let us fix node 1 as the source ($s = 1$) and nodes 2 and 3 are connected with sink node r. The resulting network is presented in Figure 1.(b).

In the term of above notations the flow model of NDP2EM can be formulated as the following integer linear program:

$$\min \sum_{e \in E} c_e x_e \tag{1}$$

subject to

$$\sum_{j\in\delta^-_M(i)} x_{ji}(M) - \sum_{j\in\delta^+_M(i)} x_{ij}(M) = \begin{cases} -b & \text{if } , i = s, \\ 0 & \text{if } , i \in V \setminus \{s\}, M \in \Pi \\ b & \text{if } , i = r, \end{cases} \tag{2}$$

$$0 \le x_{ij}(M) + x_{ji}(M) \le b x_{ij}, \quad (i,j) \in E, (i,j) \notin M \in \Pi, \tag{3}$$

$$0 \le x_{tr}(M) \le 1, \quad t \in N_0, M \in \Pi, \tag{4}$$

$$0 \le x_{ij} \le 1, (i,j) = e \in E, \tag{5}$$

$$x_{ij} = 0 \vee 1, \quad (i,j) = e \in E, \tag{6}$$

where $\delta^+_M(i)$ is the set of arcs directed out of node i and $\delta^-_M(i)$ is the set of arcs directed into node i in the network $G_r = (V \cup \{r\}, E \cup \{(t,r), t \in N \setminus \{s\}\})$, after deleting a two-edges matching M from the network G.

Note that if the graph G has not any two-edges matching then the problem is the Steiner problem on the graph G. Here, it follows that the set N is a node set of any connected subgraph of G then the NDP2EM has a solution and the conversely is true, too. By this property NDP2EM differs from the network design problem that are considered in the above cited papers.

To compute the lower bound for the optimal objective function we consider the dual problem of (1)-(5)(without the constraints (6)). The dual problem can be formulated as follows:

$$\max b \sum_{M\in\Pi} (u_r(M) - u_s(M)) - \sum_{t\in N_0} \sum_{M\in\Pi} y_{tr}(M) - b \sum_{(i,j)\in E} zz_{ij} \tag{7}$$

subject to

$$u_j(M) - u_i(M) \le w_{ij}(M), (i,j) \in E, (i,j) \notin M \in \Pi, \tag{8}$$

$$\sum_{M\in\Pi} w_{ij}(M) \le cc_{ij} + zz_{ij}, (i,j) \in E, (i,j) \notin M \in \Pi, \tag{9}$$

$$u_r(M) - u_t(M) \le y_{tr}(M), t \in N_0, M \in \Pi, \tag{10}$$

$$y_{tr}(M) \ge 0, w_{ij}(M) \ge 0, z_{ij} \ge 0, \tag{11}$$

where $cc_{ij} = c_{ij}/b$, $zz_{ij} = z_{ij}/b$. In the dual model, z_{ij} are dual variables for the constraints (5).

Proposition 1 *There exists an optimal solution of (7)-(11) for which $u_r(M) - u_t(M) = y_{tr}(M)$, for all $t \in N_0$ and $M \in \Pi$.*

Proof. We suppose that $u_r(M) - u_t(M) < y_{tr}(M)$ for some t in N_0. Let $u_r(M) - u_t(M) \ge 0$ then we can set $y_{tr}(M) = u_r(M) - u_t(M)$. If $u_r(M) - u_t(M) < 0$ then we can set $u_r(M) = u_t(M)$ and $y_{tr}(M) = 0$ to increase the optimal value of the objective function (7).

Another proof of this proposition follows from the complementary slackness optimality condition such that

$$(u_r(M) - u_t(M) - y_{tr}(M))x_{tr}(M) = 0.$$

Since the capacity of the cut which separate the node r from the other nodes is equal to b, then it follows that $x_{tr}(M) = 1$ in any feasible solution, for all nodes $t \in N_0$. Therefore, we have $u_r(M) - u_t(M) = y_{tr}(M)$ for all nodes $t \in N_0$. ■

By the proposition we can write $u_r(M) = u_t(M) + y_{tr}(M)$ for all $t \in N_0$ and $M \in \Pi$. Taking into account that $b = |N_0|$, the objective function (7) can be rewritten as follows:

$$\max \sum_{M\in\Pi} (\sum_{t\in N_0} u_t(M) - bu_s(M)) - \sum_{(i,j)\in E} z_{ij} \tag{12}$$

and the constraints (10) can be eliminated from the model.

Theorem 1 *In an optimal solution of the dual problem, if $u_j(M) - u_i(M) \neq 0$ and $w_{ij}(M) > 0$ for matchings $M = M_1, M_2 \in \Pi$ so that $(i,j) \notin M_1, M_2$, then there exits an optimal solution such that*

$$u_j(M_1) - u_i(M_1) = u_j(M_1) - u_i(M_1) + u_j(M_2) - u_i(M_2), \tag{13}$$

$$u_j(M_2) - u_i(M_2) = 0 \tag{14}$$

and

$$w_{ij}(M_1) = w_{ij}(M_1) + w_{ij}(M_2), \tag{15}$$

$$w_{ij}(M_2) = 0, \tag{16}$$

for each edge $(i,j) \in E$.

Proof. Let $w_{ij}(M) > 0$ for matchings $M = M_1, M_2 \in \Pi$ such that $(i,j) \notin M_1, M_2$ and some edge $(i,j) \in E$. Let we define values of $w_{ij}(M)$ and $u_j(M)$, $u_i(M)$ for $M = M_1, M_2$. It is clear that the constraints (8) and (9) hold for the edge (i,j) after redefining values of $w_{ij}(M)$ and $u_j(M)$, $u_i(M)$. By continuing this process for the edge (i,j), we obtain $w_{ij}(M_1) > 0$ and $w_{ij}(M) = 0$ for $M \neq M_1$. By the above redefinition, we can set $u_k^*(M_1) = \sum M \in \Pi u_k(M) - u_k(M_1)$ for $k = j, i$. Since

$$\sum_{M\in\Pi} (\sum_{t\in N_0} u_t(M) - bu_s(M)) = \sum_{t\in N_0} \sum_{M\in\Pi} (u_t(M) - u_s(M)) = \sum_{t\in N_0} (u_t^* - u_s^*),$$

where $u_t^* = u_t^*(M_t)$ and $u_s^* = u_s^*(M_s)$ for M_t and M_s in Π. ■

Conclusion

To find an optimal solution of the dual problem we use standard methods of the linear programming by using CPLEX solver and it will be a lower bound for the initial problem. Then this lower bound can be using in solving 0 and 1 initial problem by standard methods type of branch and bound.

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