Symmetry Reduction and Exact Solutions of the Non-Linear Black-Scholes Equation

S. Kovalenko, O. Patsiuk

In this paper, we investigate the non-linear Black-Scholes equation and show that by an appropriate point transformation of variables the one can be reduced to such a second-order partial differential equation, which contains only the first order derivative with respect to the time variable and does not contain the required function and the mixed derivative. For the equation obtained, we study the group-theoretical properties, namely, we find the maximal algebra of invariance in Lie sense, carry out the symmetry reduction and seek for a number of exact group-invariant solutions of this equation. Using the results obtained, we get a number of exact solutions of the Black–Scholes equation.

Keywords: Black-Scholes equation, symmetry reduction, exact solution.
MSC 2010: 22E99, 35Q91, 91G80
UDC: 517.957:512.816

Introduction

The non-linear Black-Scholes equation (BSE)

$$u_t + \frac{1}{2}\sigma^2(1 + 2\rho S u_{SS})S^2 u_{SS} + r(Su_S - u) = 0, \quad \sigma, \rho > 0,\ r \geq 0, \tag{1}$$

is widely used in financial mathematics in the study of illiquid markets (see [1], [2]). It models stock option pricing when transaction costs arising in the hedging of portfolios are taken into account. In this equation, $u(t, S)$ – value of an option, t – time, S – price of the underlying asset, $\tilde{\sigma} = \sqrt{\sigma(1 + 2\rho S u_{SS})}$ – volatility function, σ – constant volatility, ρ – parameter modelling the liquidity of the market[1], r – riskless interest rate in the bank.

Due to the importance of equation (1), it is widely studied by means of the methods of mathematical and computer modelling (see [3], [4], [5], and references therein). We are interested in finding exact solutions of equation (1) by the technique of group analysis. First of all, using the notation $a = \frac{1}{2}\sigma^2$, $b = \rho\sigma^2$, $c = r$, and $x = S$, we rewrite the equation in a more convenient form:

$$u_t + ax^2 u_{xx} + bx^3 u_{xx}^2 + cxu_x - cu = 0, \quad a, b > 0,\ c \geq 0. \tag{2}$$

In what follows, we consider only the values of independent variables t, x from the domain $\mathbb{R}_+ \times \mathbb{R}_+$ (this is due to the economic sense of these variables).

[1]For $\rho = 0$ the market is perfectly liquid (and we have the linear Black-Scholes equation), whereas for ρ large a trade has a substantial impact on the transaction price. For the stock of major U.S. corporations ρ is a small parameter (of the order of 10^{-4}) [3, p. 186].

Simplifying Point Transformations of Variables

Using the point transformations of variables

$$\bar{t} = \begin{cases} t, \ c = 0, \\ ct, \ c > 0; \end{cases}$$
$$\overline{x} = \begin{cases} \log \dfrac{x}{b}, \ c = 0, \\ \log \dfrac{cx}{b} - ct, \ c > 0; \end{cases} \tag{3}$$
$$\overline{u} = \begin{cases} \dfrac{bu}{x} + \dfrac{a}{2} \log \dfrac{x}{b} - \dfrac{a^2}{4} t, \ c = 0, \\ \dfrac{bu}{cx} + \dfrac{a}{2c} \log \dfrac{cx}{b} - \dfrac{a}{2} \left(1 + \dfrac{a}{2c}\right) t, \ c > 0, \end{cases}$$

we can reduce equation (2) to the equation

$$\overline{u}_{\bar{t}} + (\overline{u}_{\overline{x}} + \overline{u}_{\overline{xx}})^2 = 0. \tag{4}$$

Afterwards, solving this equation, we omit the overlines for convenience.

So, we get an equation of the form $u_t = F(u_x, u_{xx})$. It is known [6, Subs. 12.1.1, No. 2] that the last equation admits the traveling-wave solution

$$u(t, x) = u(\xi), \quad \xi = kx + \lambda t, \tag{5}$$

where the function $u(\xi)$ is determined by the autonomous ordinary differential equation (ODE)

$$F(ku_\xi, k^2 u_{\xi\xi}) - \lambda u_\xi = 0,$$

and a more complicated solution of the form

$$u(t, x) = c_1 + c_2 t + \varphi(\xi), \quad \xi = kx + \lambda t, \tag{6}$$

where the function $\varphi(\xi)$ is determined by the autonomous ODE

$$F(k\varphi_\xi, k^2 \varphi_{\xi\xi}) - \lambda \varphi_\xi - c_2 = 0.$$

Below we find a number of solutions of equation (4), which do not belong to the types (5) or (6).

Symmetry Reduction and Exact Solutions of Equation (4)

Using the LIE program [7], we obtain that the basis of maximal algebra of invariance (MAI) of equation (4) can be chosen as follows:

$$X_1 = -\partial_x, \quad X_2 = -e^{-x}\partial_u, \quad X_3 = \partial_t, \quad X_4 = \partial_u, \quad X_5 = t\partial_t - u\partial_u.$$

Non-zero commutators of this operators are:

$$[X_1, X_2] = X_2, \quad [X_2, X_5] = -X_2, \quad [X_3, X_5] = X_3, \quad [X_4, X_5] = -X_4.$$

Hence, the algebra

$$A = \{X_1, X_2, X_3, X_4, X_5\}$$

can be decomposed into a semidirect sum of the one-dimensional algebra $\{X_5\}$ and the four-dimensional ideal $\{X_1, X_2, X_3, X_4\}$, i.e.:

$$A = \{X_5\} \Subset \{X_1, X_2, X_3, X_4\}.$$

The ideal is of the type $A_2 \oplus 2A_1$. Using this facts and executing the well-known classification algorithm [8, p. 1450], we obtain the following assertion.

Proposition 1. *The optimal system of one-dimensional subalgebras of MAI of equation (4) consists of the following ones:* $\langle X_1\rangle$, $\langle X_2\rangle$, $\langle X_3\rangle$, $\langle X_4\rangle$, $\langle X_5\rangle$, $\langle X_1 + \varepsilon X_3\rangle$, $\langle X_1 + \varepsilon X_4\rangle$, $\langle X_2 + \varepsilon X_3\rangle$, $\langle X_2 + \varepsilon X_4\rangle$, $\langle X_3 + \varepsilon X_4\rangle$, $\langle X_1 + y(\varepsilon_1 X_3 + \varepsilon_2 X_4)\rangle$, $\langle X_2 + \sin\varphi(\varepsilon_1 X_3 + \varepsilon_2 X_4)\rangle$, $\langle X_5 + zX_1\rangle$, $\langle X_5 - X_1 + \varepsilon X_2\rangle$, *where* $\varepsilon = \pm 1$, $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, $y > 0$, $z \neq 0, -1$, *and* $0 < \varphi < \frac{\pi}{2}$.

Note that the algebras $\langle X_2\rangle$, $\langle X_4\rangle$, and $\langle X_2 + \varepsilon X_4 \mid \varepsilon = \pm 1\rangle$ do not satisfy the necessary conditions for existence of the non-degenerate invariant solutions.

Next, we perform the detailed analysis of invariant solutions of equation (4), which is based on all algebras from Proposition 1, except the ones pointed out above. The results of our investigation are presented in Tables 1 and 2. Table 1 consists of ansatzes generated by the subalgebras and corresponding reduced equations.

Table 1. The symmetry reduction of equation (4)

No.	Algebra[a]	Ansatz	Reduced equation
1	$\langle X_1\rangle$	$u = \varphi(t)$	$\varphi' = 0$
2	$\langle X_3\rangle$	$u = \varphi(x)$	$\varphi'' + \varphi' = 0$
3	$\langle X_1 + \varepsilon X_3\rangle$	$u = \varphi(x + \varepsilon t)$	$(\varphi'' + \varphi')^2 + \varepsilon\varphi' = 0$
4	$\langle X_1 + \varepsilon X_4\rangle$	$u = \varphi(t) - \varepsilon x$	$\varphi' = -1$
5	$\langle X_2 + \varepsilon X_3\rangle$	$u = \varphi(x) - \varepsilon t e^{-x}$	$(\varphi'' + \varphi')^2 - \varepsilon e^{-x} = 0$
6	$\langle X_3 + \varepsilon X_4\rangle$	$u = \varphi(x) + \varepsilon t$	$(\varphi'' + \varphi')^2 + \varepsilon = 0$
7[b]	$\langle X_1 + k(X_3 + \varepsilon X_4)\rangle$	$u = \varphi(y) + \varepsilon t$	$(\varphi'' + \varphi')^2 + \frac{1}{k}\varphi' + \varepsilon = 0$
8[c]	$\langle X_2 + k(X_3 + \varepsilon X_4)\rangle$	$u = \varphi(x) + \left(\varepsilon - \frac{1}{k}e^{-x}\right)t$	$(\varphi'' + \varphi')^2 - \frac{1}{k}e^{-x} + \varepsilon = 0$
9	$\langle X_5\rangle$	$u = t^{-1}\varphi(x)$	$(\varphi'' + \varphi')^2 - \varphi = 0$
10[d]	$\langle X_5 + kX_1\rangle$	$u = t^{-1}\varphi(y)$	$(\varphi'' + \varphi')^2 + k\varphi' - \varphi = 0$
11[e]	$\langle X_5 - X_1 + \varepsilon X_2\rangle$	$u = e^{-x}(\varphi(y) - \varepsilon x)$	$(\varphi'' - \varphi' + \varepsilon)^2 - e^y\varphi' = 0$

[a]In this column, $\varepsilon = \pm 1$.
[b]In this case, $k \neq 0$; $y = x + \frac{1}{k}t$.
[c]In this case, $0 < |k| < 1$.
[d]In this case, $k \neq 0, -1$; $y = x + k\log t$.
[e]In this case, $y = x - \log t$.

Table 2. The exact group-invariant solutions of equation (4)

No.	Algebra[a]	Exact solution or first order ODE[b]
1	1	$u = c_1$
2	2	$u = c_1 + c_2 e^{-x}$
3	3	$u = c_1 - \varepsilon(x+\varepsilon t) + 4\delta c_2 e^{-\frac{1}{2}(x+\varepsilon t)} + \varepsilon c_2^2 e^{-(x+\varepsilon t)}$
4	4	$u = c_1 - t - \varepsilon x$
5	5	$u = c_1 + 4\delta e^{-\frac{x}{2}} - (t + c_2)e^{-x}$
6	6 ($\varepsilon = -1$)	$u = c_1 + c_2 e^{-x} + \delta x - t$
7[c]	7 ($\varphi' = \mathrm{const}$)	$u = c_1 + \varepsilon t + \frac{1}{2k}\left(\delta\sqrt{1-4\varepsilon k^2} - 1\right)\left(x + \frac{1}{k}t\right)$
8[d]	8 ($\varepsilon = -1$)	$u = c_1 + c_2 e^{-x} - \left(1 + \frac{1}{k}e^{-x}\right)t + \frac{\delta}{k}\Big[\left(k - \frac{1}{2}e^{-x}\right)x -$ $- 3\sqrt{k(k+e^{-x})} + \left(2k - e^{-x}\right)\log\left(\sqrt{k} + \sqrt{k+e^{-x}}\right)\Big]$
9[d]	8 ($\varepsilon = -1$)	$u = c_1 + c_2 e^{-x} - \left(1 - \frac{1}{k}e^{-x}\right)t + \frac{\delta}{k}\Big[\left(k + \frac{1}{2}e^{-x}\right)x -$ $- 3\sqrt{k(k-e^{-x})} + \left(2k + e^{-x}\right)\log\left(\sqrt{k} + \sqrt{k-e^{-x}}\right)\Big]$
10[d]	8 ($\varepsilon = 1$)	$u = c_1 + c_2 e^{-x} + \left(1 - \frac{1}{k}e^{-x}\right)t +$ $+ \frac{\delta}{k}\Big[2\left(k + \frac{1}{2}e^{-x}\right)\arctan\sqrt{\frac{1}{k}(e^{-x}-k)} - 3\sqrt{k(e^{-x}-k)}\Big]$
11[e]	7 ($\varphi' \neq \mathrm{const}$)	$w'(z) = 1 - \varepsilon k^2 \frac{z}{w(z)}$
12	9	$w'(z) = \frac{1}{w(z)} - \sqrt[3]{\frac{4}{3z}}$
13[f]	10	$w'(\varphi) = \frac{\sqrt{\varphi - k w(\varphi)}}{w(\varphi)} - 1$
14	11	$w'(z) = 1 - \varepsilon \frac{e^{-2z}}{w(z)}$

[a]In this column, the numbers of algebras from Table 1 are indicated.
[b]In this column, $\varepsilon, \delta \in \{1, -1\}$; c_1, c_2 are arbitrary real constants.
[c]In this case, $k \neq 0$, if $\varepsilon = -1$, and $0 < |k| \leq \frac{1}{2}$, if $\varepsilon = 1$.
[d]In this case, $0 < k < 1$.
[e]In this case, $k \neq 0$.
[f]In this case, $k \neq 0, -1$.

Remark 1. *Equations 6 and 8 (with $k < 0$) from Table 1 can admit real solutions, only if $\varepsilon = -1$.*

Exact solutions (or the first order ODEs, if we could not find their solutions) are given in Table 2.

Remark 2. *In Table 2:*

1. *solution 1 is trivial and can be included to solution 2;*
2. *solution 3 is the traveling-wave one, which can be obtained from (5), if we put $k = 1$, $\lambda = \varepsilon$;*
3. *solution 4 can be obtained from solution 3, if we put $c_2 = 0$;*
4. *solution 7 is of the form (6), and one can be obtained, if we put $k = 1$, $\lambda = \frac{1}{k}$, $c_2 = \varepsilon$;*
5. *the ODE 11 is obtained, if in the ODE 7 from Table 1 we put*

$$z = -\frac{1}{k}e^{\frac{1}{2}y}, \quad \omega = e^{\frac{1}{2}y}\sqrt{-\left(\varepsilon + \frac{1}{k}\varphi'(y)\right)},$$

and admits the solution in the parametric form [9, Subs. 1.3.1, No. 2]: $z = z(\tau)$, $w = \tau \cdot z(\tau)$, *where* $z(\tau)$ *is defined as:*

$$(a)\ z(\tau) = c_1 \left(\left|2\tau - 1 + \sqrt{4k^2+1}\right|^{1-\frac{1}{\sqrt{4k^2+1}}} \right.$$
$$\left. \left|2\tau - 1 - \sqrt{4k^2+1}\right|^{1+\frac{1}{\sqrt{4k^2+1}}} \right)^{-\frac{1}{2}},$$

if $\varepsilon = -1$ *and* $k \neq 0$*;*

$$(b)\ z(\tau) = c_1 \left(\left|2\tau - 1 + \sqrt{1-4k^2}\right|^{1-\frac{1}{\sqrt{1-4k^2}}} \right.$$
$$\left. \left|2\tau - 1 - \sqrt{1-4k^2}\right|^{1+\frac{1}{\sqrt{1-4k^2}}} \right)^{-\frac{1}{2}},$$

if $\varepsilon = 1$ *and* $0 < |k| < \frac{1}{2}$*;*

$$(c)\ z(\tau) = \frac{c_1}{2\tau - 1}\, e^{\frac{1}{2\tau-1}},$$

if $\varepsilon = 1$ *and* $k = \pm\frac{1}{2}$*;*

$$(d)\ z(\tau) = c_1(\tau^2 - \tau + k^2)^{-\frac{1}{2}}\, e^{-\frac{1}{\sqrt{4k^2-1}} \arctan \frac{2\tau-1}{\sqrt{4k^2-1}}},$$

if $\varepsilon = 1$ *and* $|k| > \frac{1}{2}$*;*

6. *the ODE 12 is obtained, if in the ODE 9 from Table 1 we put*

$$z = \frac{1}{6}\sqrt{\varphi^3},\ \omega = \frac{1}{2}\varphi';$$

   note that this equation is the Abel equation of the second kind;
7. *the ODE 13 is obtained, if in the ODE 10 from Table 1 we put* $w(\varphi) = \varphi'$*;*
8. *the ODE 14 is obtained, if in the ODE 11 from Table 1 we put*

$$z = \frac{1}{2}y, \quad \omega = e^{-\frac{1}{2}y}\sqrt{\varphi'(y)}.$$

Exact Solutions of Equation (2)

Using solutions 2–3, and 5–10 of equation (4) (see Table 2) and point transformations of variables (3), we obtain a number of exact solutions of equation (2) presented in Tables 3 and 4 (see the next pages).

Remark 3. *Solutions 7 and 8 from Tables 3 and 4 are not defined for all values of* $x \in \mathbb{R}_+$*. Thus, they cannot be considered as the solutions of any boundary value problem (BVP) determined for equation (2) on the domain* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$*.*

Table 3. The exact solutions of equation (2) with $c = 0$

No.	Sol.[a]	Exact solution[b]
1	2	$u = c_1 + \frac{a}{2b}x\left(c_2 + \frac{a}{2}t - \log x\right)$
2	5	$u = c_1 - t + 4\delta\sqrt{\frac{x}{b}} + \frac{a}{2b}x\left(c_2 + \frac{a}{2}t - \log x\right)$
3	6	$u = c_1 + \frac{a+2\varepsilon}{2b}x\left(c_2 + \frac{a-2\varepsilon}{2b}t - \log x\right)$
4	3	$u = \varepsilon c_1^2 e^{-\varepsilon t} + 4\delta c_1 e^{-\frac{\varepsilon}{2}t}\sqrt{\frac{x}{b}} + \frac{a+2\varepsilon}{2b}x\left(c_2 + \frac{a-2\varepsilon}{2}t - \log x\right)$
5[c]	7	$u = \frac{x}{b}\left[c_1 + \left(\varepsilon + \frac{a^2}{4}\right)t - \frac{a}{2}\log x + \frac{1}{2k}\left(\delta\sqrt{1-4\varepsilon k^2} - 1\right)\left(\frac{1}{k}t + \log x\right)\right]$
6[d]	8	$u = c_1 - \frac{1}{k}t + \frac{x}{b}\left\{c_2 + \left(\frac{a^2}{4} - 1\right)t - \frac{a}{2}\log x + \right.$ $\left. + \delta\left[\left(1 - \frac{b}{2kx}\right)\log\left(\frac{2kx}{b}\left(1 + \sqrt{1 + \frac{b}{kx}}\right) + 1\right) - 3\sqrt{1 + \frac{b}{kx}}\right]\right\}$
7[d]	9	$u = c_1 + \frac{1}{k}t + \frac{x}{b}\left\{c_2 + \left(\frac{a^2}{4} - 1\right)t - \frac{a}{2}\log x + \right.$ $\left. + \delta\left[\left(1 + \frac{b}{2kx}\right)\log\left(\frac{2kx}{b}\left(1 + \sqrt{1 - \frac{b}{kx}}\right) - 1\right) - 3\sqrt{1 - \frac{b}{kx}}\right]\right\}$
8[d]	10	$u = c_1 - \frac{1}{k}t + \frac{x}{b}\left\{c_2 + \left(\frac{a^2}{4} + 1\right)t - \frac{a}{2}\log x + \right.$ $\left. + \delta\left[2\left(1 + \frac{b}{2kx}\right)\arctan\sqrt{\frac{b}{kx} - 1} - 3\sqrt{\frac{b}{kx} - 1}\right]\right\}$

[a]In this column, the numbers of solutions of equation (4) from Table 2 are indicated.
[b]In this column, $\varepsilon, \delta \in \{1, -1\}$; c_1, c_2 are arbitrary real constants.
[c]In this case, $k \neq 0$, if $\varepsilon = -1$, and $0 < |k| \leq \frac{1}{2}$, if $\varepsilon = 1$.
[d]In this case, $0 < k < 1$.

Table 4. The exact solutions of equation (2) with $c \neq 0$

No.	S.[a]	Exact solution[b]
1	2	$u = c_1 e^{ct} + \frac{a}{2b}x\left(c_2 + \frac{a+2c}{2}t - \log x\right)$
2	5	$u = (c_1 - ct)e^{ct} + 4\delta e^{\frac{c}{2}t}\sqrt{\frac{cx}{b}} + \frac{a}{2b}x\left(c_2 + \frac{a+2c}{2}t - \log x\right)$
3	6	$u = c_1 e^{ct} + \frac{a+2\varepsilon c}{2b}x\left(c_2 + \frac{a+2(1-\varepsilon)c}{2}t - \log x\right)$
4	3	$u = \varepsilon c_1^2 e^{(1-\varepsilon)ct} + 4\delta c_1 e^{\frac{c}{2}(1-\varepsilon)t}\sqrt{\frac{cx}{b}} + \frac{a+2\varepsilon c}{2b}x\left(c_2 + \frac{a+2(1-\varepsilon)c}{2}t - \log x\right)$
5[c]	7	$u = \frac{cx}{b}\left\{c_1 + \left[\varepsilon c + \frac{a}{2}\left(1 + \frac{a}{2c}\right)\right]t - \frac{a}{2c}\log x + \right.$ $\left. + \frac{1}{2k}\left(\delta\sqrt{1-4\varepsilon k^2} - 1\right)\left[\left(\frac{1}{k} - 1\right)ct + \log x\right]\right\}$
6[d]	8	$u = \left(c_1 - \frac{c}{k}t\right)e^{ct} + \frac{cx}{b}\left\{c_2 + \left[\frac{a}{2}\left(1 + \frac{a}{2c}\right) - c\right]t - \frac{a}{2c}\log x + \right.$ $\left. + \delta\left[\left(1 - \frac{b}{2kcx}e^{ct}\right)\log\left(\frac{2kcx}{b}e^{-ct}\left(1 + \sqrt{1 + \frac{b}{kcx}e^{ct}}\right) + 1\right) - 3\sqrt{1 + \frac{b}{kcx}e^{ct}}\right]\right\}$
7[d]	9	$u = \left(c_1 + \frac{c}{k}t\right)e^{ct} + \frac{cx}{b}\left\{c_2 + \left[\frac{a}{2}\left(1 + \frac{a}{2c}\right) - c\right]t - \frac{a}{2c}\log x + \right.$ $\left. + \delta\left[\left(1 + \frac{b}{2kcx}e^{ct}\right)\log\left(\frac{2kcx}{b}e^{-ct}\left(1 + \sqrt{1 - \frac{b}{kcx}e^{ct}}\right) - 1\right) - 3\sqrt{1 - \frac{b}{kcx}e^{ct}}\right]\right\}$
8[d]	10	$u = \left(c_1 - \frac{c}{k}t\right)e^{ct} + \frac{cx}{b}\left\{c_2 + \left[\frac{a}{2}\left(1 + \frac{a}{2c}\right) + c\right]t - \frac{a}{2c}\log\frac{cx}{b} + \right.$ $\left. + \delta\left[2\left(1 + \frac{b}{2kcx}e^{ct}\right)\arctan\sqrt{\frac{b}{kcx}e^{ct} - 1} - 3\sqrt{\frac{b}{kcx}e^{ct} - 1}\right]\right\}$

[a]In this column, the numbers of solutions of equation (4) from Table 2 are indicated.
[b]In this column, $\varepsilon, \delta \in \{1, -1\}$; c_1, c_2 are arbitrary real constants.
[c]In this case, $k \neq 0$, if $\varepsilon = -1$, and $0 < |k| \leq \frac{1}{2}$, if $\varepsilon = 1$.
[d]In this case, $0 < k < 1$.

Conclusions

In this paper, we found a number of exact group-invariant solutions of the nonlinear Black-Scholes equation (2). These solutions can be used in solving some BVPs. In our future investigations, we are going to consider from the group-theoretical point of view a BVP for the European options.

References

[1] M. Avellaneda, A. Levy, and A. Páras, "Pricing and hedging derivative securities in markets with uncertain volatilities," *Appl. Math. Finance*, vol. 2, no. 2, pp. 73–88, 1995.

[2] U. Çetin, M. Soner, and N. Touzi, "Option hedging for small investors under liquidity cost," *Finance Stoch.*, vol. 14, no. 3, pp. 317–341, 2010.

[3] R. Frey and U. Polte, "Nonlinear black–scholes equation in finance: associated control problems and properties of solutions," *SIAM J. Control Optim.*, vol. 49, no. 1, pp. 185–204, 2011.

[4] R. Company, L. Jódar, and J.-R. Pintos, "A numerical method for european option pricing with transaction costs nonlinear equation," *Math. Comput. Model.*, vol. 50, no. 5–6, pp. 910–920, 2009.

[5] J. Ankudinova and M. Ehrhardt, "On the numerical solution of nonlinear black–scholes equations," *Comput. Math. Appl.*, vol. 56, no. 3, pp. 799–812, 2008.

[6] A. D. Polyanin and V. F. Zaitsev, *Handbook of nonlinear partial differential equations.* CRC Press, 2012.

[7] A. K. Head, "Lie, a pc program for lie analysis of differential equations," *Comput. Phys. Commun.*, vol. 96, no. 2–3, pp. 311–313, 1996.

[8] J. Patera and P. Winternitz, "Subalgebras of real three- and four-dimensional lie algebras," *Finance Stoch.*, vol. 18, no. 7, pp. 1449–1455, 1977.

[9] A. D. Polyanin and V. F. Zaitsev, *Handbook of exact solutions for ordinary differential equations.* CRC Press, 2003.

Authors

Sergii Kovalenko — Senior Lecturer, Department of Physics, Poltava National Technical University, Poltava, Ukraine; E-mail: *kovalenko@imath.kiev.ua*

Oleksii Patsiuk — Junior Researcher, Department of Partial Differential Equations, Institute of Mathematics of National Academy of Sciences of Ukraine, Kyiv, Ukraine; E-mail: *patsyuck@yahoo.com*